Macroeconomics 1 (4/7)

The growth model with learning by doing (Romer, 1986)

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September – December 2024

Key features of the model

- Romer's (1986) model endogenizes the technological progress of the Cass-Koopmans-Ramsey model in order to better explain long-term growth.

- **Paul M. Romer**: American economist, born in 1955 in Denver, professor at New York University since 2011, co-laureate (with William D. Nordhaus) of the Sveriges Riksbank's prize in economic sciences in memory of Alfred Nobel in 2018 "*for integrating technological innovations into long-run macroeconomic analysis*".

- This model rests on two key concepts:
  - **learning by doing**,
  - **knowledge diffusion**.

- It endogenizes the saving rate of Frankel's (1962) model in the same way as the Cass-Koopmans-Ramsey model endogenizes the saving rate of the Solow-Swan model.

Private and social returns of capital

- This model distinguishes between
  - the **private** returns of capital, which are **strictly decreasing**,
  - the **social** returns of capital, which are **constant**.

- It is an "**AK model**" $\equiv$ model in which the aggregate production function can be written in a form of type $Y_t = \mathcal{A}_t K_t$ where $\mathcal{A}_t$ is exogenous (be careful not to confuse $\mathcal{A}_t$ with A_t).

- The constant social returns of capital will
  - generate long-term growth,
  - imply no conditional convergence.

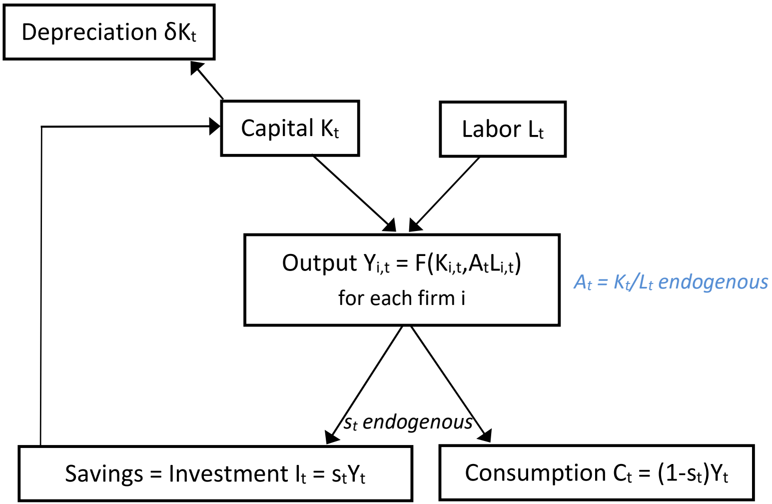
- The gap between the private and social returns of capital will give a role to economic policy.

General overview of the model I *

- Each firm rents capital and employs labor to produce goods, with a **labor effectiveness** depending on **aggregate capital** (stock).
- Households own capital and supply labor.
- The goods produced by firms are used for households' consumption and investment in new capital.
- The saving rate is endogenous, optimally chosen by households.
- Capital evolves over time due to investment and capital depreciation.

(In the pages whose title is followed by an asterisk,
in blue: changes from Chapter 2.)

General overview of the model II *



Exogenous variables *

- **Neither flows nor stocks**:
    - continuous time, indexed by t,
    - price of goods $\equiv$ numéraire $= 1$,
    - (large) number of firms I.

- **Flow**:
    - labor supply $= 1$ per person.

- **Stocks**:
    - agregate initial capital $K_0 > 0$,
    - population $L_t = L_0 e^{nt}$, where $L_0 > 0$ and $n \geq 0$,
    - ~~productivity parameter $A_t = A_0 e^{gt}$, where $A_0 > 0$ and $g \geq 0$.~~

Endogenous variables *

- **Prices**:
  - real usage cost of capital z_t,
  - real wage w_t,
  - real interest rate r_t.
- **Quantities – flows**:
  - output $Y_{i,t}$ of firm i,
  - labor demand $N_{i,t}$ of firm i,
  - aggregate output $Y_t \equiv \sum_{i=1}^{I} Y_{i,t}$,
  - aggregate labor demand $N_t \equiv \sum_{i=1}^{I} N_{i,t}$,
  - aggregate consumption C_t.
- **Quantities – stocks**:
  - capital $K_{i,t}$ of firm i (except at $t = 0$),
  - aggregate capital $K_t \equiv \sum_{i=1}^{I} K_{i,t}$ (except at $t = 0$),
  - real aggregate amount of assets B_t,
  - productivity parameter A_t.

Good, private agents, markets, general-equil. conditions *

- The good, private agents, and markets are the same as in Chapter 2. In particular, markets are **perfectly competitive**.

- Each private agent solves their optimization problem: as all markets are perfectly competitive,
  - at each time $t \geq 0$, each firm i chooses $(Y_{i,t}, K_{i,t}, N_{i,t})$, as a function of the prices (w_t, z_t, r_t) and of productivity A_t that they consider as given, in order to maximize their *instantananeous* profit,
  - at time 0, the representative household chooses $(\frac{C_t}{L_t}, \frac{B_t}{L_t})_{t\geq 0}$, as a function of the prices $(w_t, z_t, r_t)_{t\geq 0}$ that they consider as given, in order to maximize their *intertemporal* utility (under perfect expectations) subject to constraints.

- Prices are such that each market is cleared at each time $t \geq 0$:
  - w_t clears the labor market: $N_t = L_t$,
  - z_t clears the capital market,
  - r_t clears the loan market.

Chapter outline

Equilibrium conditions

Households' behavior *

- Households are modeled exactly as in Chapter 2, with a constant elasticity of intertemporal substitution, equal to $\frac{1}{\theta}$.

- Their behavior is thus characterized by the equilibrium conditions
  - $\dot{b}_t = w_t + (r_t - n)b_t - c_t$ (**instantaneous budget constraint**),
  - $\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta}$ (**Euler equation**),
  - $\lim_{t \to +\infty} \left[b_t e^{-\int_0^t (r_\tau - n) d\tau} \right] = 0$ (**transversality condition**),

  where
  - $c_t \equiv \frac{C_t}{L_t}$ is per-capita consumption,
  - ρ is the rate of time preference ($\rho > n > 0$),
  - $b_t \equiv \frac{B_t}{L_t}$ is the aggregate amount of assets in units of goods per person.

Production function and labor effectiveness

- Output of each firm i: $Y_{i,t} = F(K_{i,t}, A_t N_{i,t})$, where the production function F has the same properties as in Chapters 1 and 2.

- Labor effectiveness in each firm i: $A_t = \frac{K_t}{L_t}$ (and not $A_{i,t} = \frac{K_{i,t}}{N_{i,t}}$).

- This specification captures two concepts defined by Arrow (1962):
  - **learning by doing**: the larger the per-capita stock of capital (which reflects the accumulated past per-capita production and thus the experience of each worker), the more effective each worker;
  - **knowledge diffusion** (assumed to be instantaneous) across firms, because of the non-rival and non-excludable nature of knowledge (which explains why the effectiveness of workers in firm i depends on K_t / L_t, not $K_{i,t} / N_{i,t}$).

Non-rivalry and non-excludability

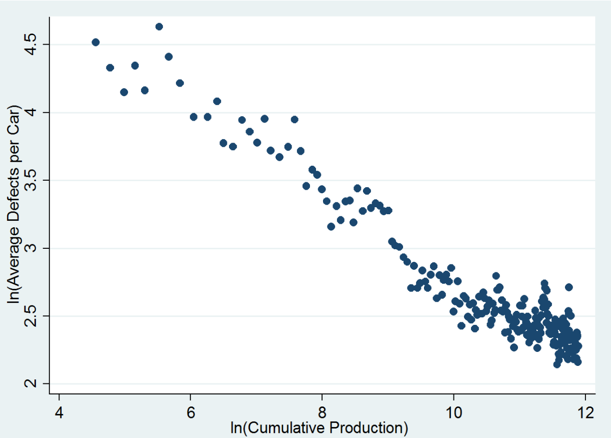
- **Non-rival** good $\equiv$ good whose consumption by an agent has no effect on the quantity available for other agents.
- **Non-excludable** good $\equiv$ good from which each agent can benefit costlessly.
- In Chapter 5, we will consider a non-rival but excludable good (namely, the ability or the right to produce a type of intermediate good, due to a trade secret or a patent).

Labor effectiveness

- Labor effectiveness $A_t = \frac{K_t}{L_t}$ is a stock.
- This captures the idea that knowledge and know-how accumulate over time.
- **Kenneth J. Arrow**: American economist, born in 1921 in New York, deceased in 2017 in Palo Alto, professor at Stanford University from 1979, co-laureate (with John R. Hicks) of the Sveriges Riksbank's prize in economic sciences in memory of Alfred Nobel in 1972 "*for their pioneering contributions to general economic equilibrium theory and welfare theory*".

An example of learning by doing

Log of the average number of defects per car as a function of the log of the cumulative number of cars produced (in a car factory)



Source: Levitt, List and Syverson (2013).

Firms' optimization problem *

- As in Chapter 2, we assume that
  - firms rent their capital stock at each time,
  - there is no capital-adjustment cost.
- So, at each time t, firm i chooses $K_{i,t}$ and $N_{i,t}$ to maximize their *instantaneous* profit

$$F(K_{i,t}, A_t N_{i,t}) - z_t K_{i,t} - w_t N_{i,t}$$

taking z_t, w_t and $A_t = \frac{K_t}{L_t}$ as given.

First-order conditions *

- As in Chapter 2, denoting by F_j the partial derivative of F with respect to its j^{th} argument for $j \in \{1, 2\}$, we get the first-order conditions

$$F_1(K_{i,t}, A_t N_{i,t}) = z_t \quad \text{(marginal productivity of capital = usage cost)},$$
$$A_t F_2(K_{i,t}, A_t N_{i,t}) = w_t \quad \text{(marginal productivity of labor = wage)}.$$

- As in Chapter 2, we deduce that
  - the instantaneous profit is zero for any $K_{i,t}$ and $N_{i,t}$,
  - $\frac{K_{i,t}}{N_{i,t}}$ does not depend on i and is therefore equal to $\frac{K_t}{N_t}$,
  - $Y_t \equiv \sum_{i=1}^{I} Y_{i,t} = F(K_t, A_t N_t)$.

Social returns of capital

- Using $A_t = \frac{K_t}{L_t}$, we then get the *aggregate* production function

$$Y_t = K_t F\left(1, \frac{N_t}{L_t}\right) \equiv F^S\left(K_t, \frac{N_t}{L_t}\right).$$

- Denoting by $F^S_{j,j}$ the second derivative of F^S with respect to its j^{th} argument for $j \in \{1, 2\}$, we get

$$\forall K_t > 0, \qquad F^S_{1,1}\left(K_t, \frac{N_t}{L_t}\right) = 0,$$

so **the social returns of capital are constant**.

Private returns of capital

- The *individual* production function of firm i is

$$Y_{i,t} = F\left(K_{i,t}, \frac{K_t}{L_t} N_{i,t}\right) \equiv F^P\left(K_{i,t}, N_{i,t}, \frac{K_t}{L_t}\right).$$

- Denoting by $F_{j,j}^P$ the second derivative of F^P with respect to its j^{th} argument for $j \in \{1, 2, 3\}$, we get

$$\forall K_{i,t} > 0, \qquad F_{1,1}^P\left(K_{i,t}, N_{i,t}, \frac{K_t}{L_t}\right) < 0,$$

so **the private returns of capital are strictly decreasing**.

Usage cost of capital *

- As in Chapter 2, we assume that capital depreciates at rate δ.
- As in Chapter 2, we assume that households can
  - rent their goods as capital to firms,
  - lend their goods to other households.
- So, as in Chapter 2, we get the equilibrium condition

$$r_t = z_t - \delta.$$

Market clearing *

- As in Chapter 2, the market-clearing conditions are
  - $B_t = K_t$ (asset markets),
  - $N_t = L_t$ (labor market),
  - $\dot{K}_t = Y_t - C_t - \delta K_t$ (goods market).

- Using $N_t = L_t$, we can rewrite the aggregate production function as $Y_t = F(1, 1)K_t$, so the model is an AK model.

Equilibrium determination

1. Introduction
2. Equilibrium conditions
3. Equilibrium determination
   - Equilibrium conditions on k_t and c_t
   - Determination of k_t and c_t
   - Implications
4. Equilibrium sub-optimality
5. Conclusion
6. Appendix

Equilibrium conditions on k_t and c_t I *

- Defining $f(x) \equiv F(x,1)$ for any $x > 0$ and differentiating $F(K_{i,t}, A_t N_{i,t}) = A_t N_{i,t} f(\frac{K_{i,t}}{A_t N_{i,t}})$ with respect to $K_{i,t}$ and $N_{i,t}$, we get

$$F_1(K_{i,t}, A_t N_{i,t}) = f'\left(\frac{K_{i,t}}{A_t N_{i,t}}\right),$$

$$A_t F_2(K_{i,t}, A_t N_{i,t}) = A_t\left[f\left(\frac{K_{i,t}}{A_t N_{i,t}}\right) - \frac{K_{i,t}}{A_t N_{i,t}} f'\left(\frac{K_{i,t}}{A_t N_{i,t}}\right)\right].$$

- Using $\frac{K_{i,t}}{N_{i,t}} = \frac{K_t}{N_t}$, $N_t = L_t$, $A_t = \frac{K_t}{L_t} \equiv k_t$ and $r_t = z_t - \delta$, we can then rewrite the first-order conditions of firms' optimization problem as

$$r_t = f'(1) - \delta \text{ and } w_t = [f(1) - f'(1)]k_t.$$

Equilibrium conditions on k_t and c_t II

- The last conditions enable us to rewrite households' instantaneous budget constraint as

$$\dot{b}_t = [f(1) - f'(1)]k_t + [f'(1) - (n+\delta)]b_t - c_t.$$

- Using $B_t = K_t$, which implies $b_t = k_t$, we then get

$$\dot{k}_t = f(1)k_t - c_t - (n+\delta)k_t.$$

- This differential equation can be interpreted as "variation in the capital stock = savings − dilution − dépréciation" (per effective-labor unit) and is nothing else than the goods-market-clearing condition (consequence of Walras' law).

Equilibrium conditions on k_t and c_t III

- Using $r_t = f'(1) - \delta$, we can rewrite the Euler equation as

$$\frac{\dot{c}_t}{c_t} = \frac{f'(1) - (\delta + \rho)}{\theta}.$$

- Using $b_t = k_t$ and $r_t = f'(1) - \delta$, we can rewrite the transversality condition as

$$\lim_{t \to +\infty} \left\{ k_t e^{-[f'(1) - (n+\delta)]t} \right\} = 0.$$

Equilibrium conditions on k_t and c_t IV *

- $(k_t)_{t\geq 0}$ and $(c_t)_{t\geq 0}$ are therefore determined by two differential equations, one initial condition and one terminal condition:

$$\dot{k}_t = [f(1) - (n+\delta)]k_t - c_t,$$

$$\frac{\dot{c}_t}{c_t} = \frac{f'(1) - (\delta + \rho)}{\theta},$$

$$k_0 = \frac{K_0}{L_0},$$

$$\lim_{t \to +\infty} \left\{ k_t e^{-[f'(1) - (n+\delta)]t} \right\} = 0.$$

- The other endogenous variables are residually determined, from $(k_t)_{t\geq 0}$ and $(c_t)_{t\geq 0}$, using the other equilibrium conditions.

Determination of k_t and c_t I

- Integrating the differential equation in $\dot{c}_t$, we get

$$c_t = c_0 e^{\frac{f'(1)-(\delta+\rho)}{\theta}t}.$$

- We restrict the analysis to parameter values such that
  - $f'(1) > \delta + \rho$, for the growth rate of per-capita consumption to be positive,
  - $\rho - n > \frac{1-\theta}{\theta}[f'(1) - (\delta + \rho)]$, for intertemporal utility to take a finite value.

Determination of k_t and c_t II

- We can then rewrite the differential equation in $\dot{k}_t$ as

$$\dot{k}_t = [f(1) - (n+\delta)]k_t - c_0 e^{\frac{f'(1)-(\delta+\rho)}{\theta} t}.$$

- Then, rearranging the terms and multiplying by $e^{-[f(1)-(n+\delta)]t}$,

$$\left\{\dot{k}_t - [f(1) - (n+\delta)]k_t\right\} e^{-[f(1)-(n+\delta)]t} = -c_0 e^{-\varphi t},$$

where $\varphi \equiv f(1) - (n+\delta) - \frac{f'(1)-(\delta+\rho)}{\theta}$.

- We show in the appendix that $\varphi > f(1) - f'(1) > 0$.

Determination of k_t and c_t III

- We can therefore integrate the previous equality to get

$$k_t e^{-[f(1)-(n+\delta)]t} - k_0 = \frac{c_0}{\varphi} e^{-\varphi t} - \frac{c_0}{\varphi}$$

$$\text{and then } k_t = \left(k_0 - \frac{c_0}{\varphi}\right) e^{[f(1)-(n+\delta)]t} + \frac{c_0}{\varphi} e^{\frac{f'(1)-(\delta+\rho)}{\theta} t}.$$

- The transversality condition can then be rewritten as

$$\lim_{t \to +\infty} \left\{ \left(k_0 - \frac{c_0}{\varphi}\right) e^{[f(1)-f'(1)]t} + \frac{c_0}{\varphi} e^{[f(1)-f'(1)-\varphi]t} \right\} = 0$$

and implies $c_0 = \varphi k_0 > 0$ since $\varphi > f(1) - f'(1) > 0$ (as in Chapter 2, c_0 adjusts to satisfy the transversality condition).

Determination of k_t and c_t IV

- We therefore finally obtain

$$k_t = k_0 e^{\frac{f'(1)-(\delta+\rho)}{\theta}t} \text{ and } c_t = \varphi k_0 e^{\frac{f'(1)-(\delta+\rho)}{\theta}t}.$$

- So,
  - **the per-capita stock of capital k_t,**
  - **per-capita consumption c_t,**
  - **per-capita output $y_t = f(1)k_t$**

  grow at the same constant rate.

- This growth rate, equal to $\frac{f'(1)-(\delta+\rho)}{\theta}$, depends
  - positively on $f'(1)$ and $\frac{1}{\theta}$,
  - negatively on δ and ρ,

  which can be interpreted with the Euler equation, as in Chapter 2.

Determination of k_t and c_t V

- Because of the **constant** social returns of capital,
  - the long-term growth rate depends on $f'(1)$, $\frac{1}{\theta}$, δ and ρ,
  - the convergence to the steady state is instantaneous,

  which is not the case in the Cass-Koopmans-Ramsey model, in which the returns of capital are **decreasing**.

- The initial level of per-capita consumption $c_0 = \varphi k_0$ depends
  - positively on k_0, $f(1)$, ρ and (if $\frac{1}{\theta} > 1$) δ,
  - negatively on $f'(1)$, n, $\frac{1}{\theta}$ and (if $\frac{1}{\theta} < 1$) δ.

- c_0 and $\frac{\dot{c}_t}{c_t}$ react in opposite ways to a variation in ρ, $f'(1)$, $\frac{1}{\theta}$ or (if $\frac{1}{\theta} > 1$) δ in order to satisfy the intertemporal budget constraint.

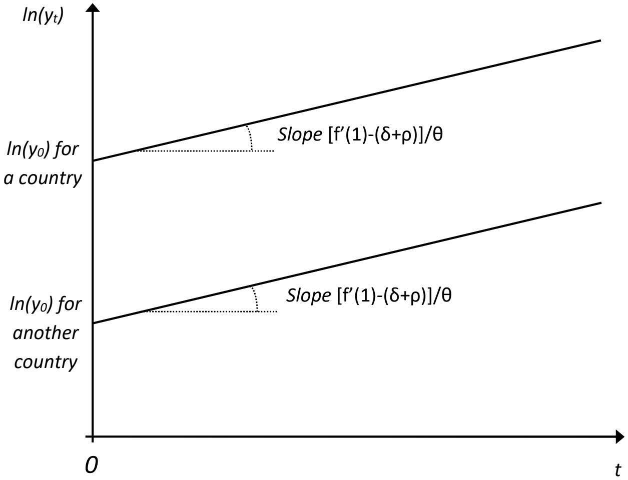
Stylised facts of Kaldor (1961)

- Romer's (1986) model thus accounts not only for the first five **stylised facts of Kaldor (1961)**, as the Cass- Koopmans-Ramsey model at the steady state, but also for the 6th one:
  1. per-capita output grows: $\frac{\dot{y}_t}{y_t} = \frac{f'(1)-(\delta+\rho)}{\theta} \geq 0$,
  2. the per-capita capital stock grows: $\frac{\dot{k}_t}{k_t} = \frac{f'(1)-(\delta+\rho)}{\theta} \geq 0$,
  3. the rate of return of capital is constant: $r_t = f'(1) - \delta$,
  4. the ratio capital / output is constant: $\frac{K_t}{Y_t} = \frac{1}{f(1)}$,
  5. the labor and capital shares of income are constant: $\frac{w_t L_t}{Y_t} = \frac{f(1)-f'(1)}{f(1)}$ and $\frac{z_t K_t}{Y_t} = \frac{f'(1)}{f(1)}$,
  6. the growth rate of per-capita output varies across countries: $\frac{\dot{y}_t}{y_t} = \frac{f'(1)-(\delta+\rho)}{\theta}$ varies across countries when the preference parameters ρ and θ vary across countries.

Neither absolute convergence, nor conditional convergence

- We have $\ln(y_t) = \ln(y_0) + \frac{f'(1)-(\delta+\rho)}{\theta} t$, where $y_0 = f(1)k_0$.
- There is therefore no long-term convergence of $\ln(y_t)$ across countries that have different y_0s, even if they have the same
  - production function $f(.)$,
  - parameters governing the dynamics of capital and labor n, δ,
  - preference parameters ρ, θ.
- The model therefore predicts **no absolute convergence and no conditional convergence** of $\ln(y_t)$ across countries, unlike the Solow-Swan and Cass-Koopmans-Ramsey models.
- The absence of conditional convergence is not supported by empirical evidence, as seen in Chapter 1.

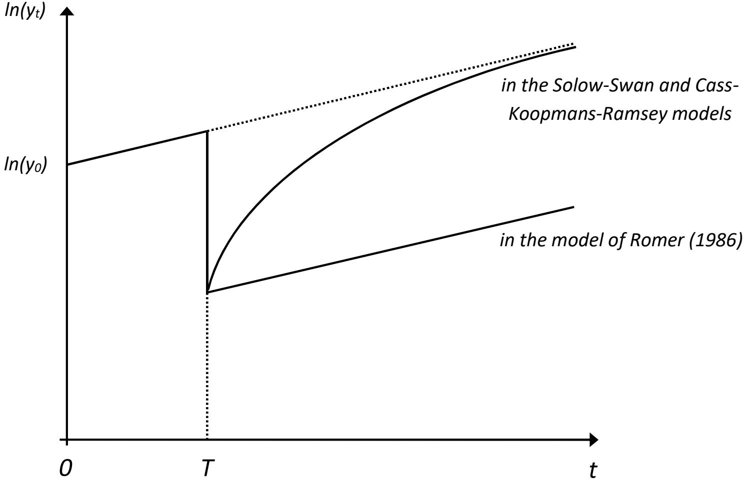
No conditional convergence



Permanent effect of shocks

- An unexpected exogenous shock on the capital stock does not modify the slope of the path of $\ln(y_t)$, but modifies its y-intercept.
- So, following such a shock, $\ln(y_t)$ does not "catch up" its initial path: **the shock has a permanent effect**.
- This prediction is consistent with the hypothesis, not rejected in the data, of unitary roots in macroeconomic time series.
- The Solow-Swan and Cass-Koopmans-Ramsey models predict on the contrary that such a shock has no permanent effect on $\ln(y_t)$ because it does not affect the steady-state path of $\ln(y_t)$.

Effect of an unexpected negative shock on capital at T



Equilibrium sub-optimality

Externality I

- For some given $(K_{j,t})_{j \neq i}$, a variation in $K_{i,t}$ has both
  - a direct effect on $Y_{i,t} = F(K_{i,t}, A_t N_{i,t})$,
  - an indirect effect on all the $Y_{j,t}$ for $j \in \{1, ..., I\}$, via $A_t = \frac{K_t}{L_t}$.
- Firm i takes only the first effect into account when choosing $K_{i,t}$ because
  - it does not take into account the indirect effect on the $Y_{j,t}$ for $j \neq i$,
  - the indirect effect on $Y_{i,t}$ is negligible compared to the direct effect on $Y_{i,t}$ (I being large, a variation in $K_{i,t}$ has little effect on K_t and A_t).
- We say that there is a **knowledge-diffusion externality** between firms.

Externality II

- A variation in K_t two simultaneous effects on $Y_t = F(K_t, A_t N_t)$:
  - a direct effect,
  - an indirect effect, via $A_t = \frac{K_t}{L_t}$.
- The benevolent, omniscient and omnipotent planner $\mathcal{BOOP}$ takes these two effects into account when choosing K_t, as they are of the same order of magnitude.
- We say that **the** $\mathcal{BOOP}$ **internalizes the** knowledge-diffusion **externality** between firms.
- We should therefore expect that, compared to the competitive equilibrium, the $\mathcal{BOOP}$ will order more investment.

Social sub-optimality of the competitive equilibrium I

- The competitive equilibrium is socially optimal if and only if it coincides with the allocation chosen by the $\mathcal{BOOP}$.
- Optimization problem of the $\mathcal{BOOP}$: for a given $k_0 > 0$,

$$\max_{(c_t)_{t\geq 0},(k_t)_{t>0}} \int_0^{+\infty} e^{-(\rho-n)t} \left(\frac{c_t^{1-\theta}-1}{1-\theta} \right) dt$$

subject to the constraints

1. $\forall t \geq 0$, $c_t \geq 0$ (non-negativity of consumption),
2. $\forall t > 0$, $k_t \geq 0$ (non-negativity of capital),
3. $\forall t \geq 0$, $\dot{k}_t = [f(1) - (n+\delta)]k_t - c_t$ (technology and resource constraint).

Social sub-optimality of the competitive equilibrium II

- **Hamiltonian** associated with the optimization problem of the $\mathcal{BOOP}$:

$$H^p(c_t, k_t, \lambda_t^p, t) \equiv e^{-(\rho-n)t}\left(\frac{c_t^{1-\theta}-1}{1-\theta}\right) + \lambda_t^p\{[f(1)-(n+\delta)]k_t - c_t\}$$

where λ_t^p represents the value, measured in utility units at time 0, of an increase of one unit of good in the resources at time t.

- Applying the optimal-control theory, we then get
  - $\lambda_t^p = e^{-(\rho-n)t}c_t^{-\theta}$ (first-order condition on the control variable),
  - $\dot{\lambda}_t^p = [n+\delta-f(1)]\lambda_t^p$ (costate equation),
  - $\dot{k}_t = [f(1)-(n+\delta)]k_t - c_t$ (resource constraint),
  - $\lim_{t\to+\infty} k_t\lambda_t^p = 0$ (transversality condition).

Social sub-optimality of the competitive equilibrium III

- Manipulating these conditions in the same way as in Chapter 2, we get
  - $\dot{k}_t = [f(1) - (n+\delta)]k_t - c_t$ (differential equation in $\dot{k}_t$),
  - $\frac{\dot{c}_t}{c_t} = \frac{f(1)-(\rho+\delta)}{\theta}$ (differential equation in $\dot{c}_t$),
  - $\lim_{t \to +\infty} \left\{ k_t e^{-[f(1)-(n+\delta)]t} \right\} = 0$ (transversality condition).
- These three conditions and $k_0 = \frac{K_0}{L_0}$ determine $(k_t)_{t \geq 0}$ and $(c_t)_{t \geq 0}$.
- We integrate the differential equation in $\dot{c}_t$ and get $c_t = c_0 e^{\frac{f(1)-(\delta+\rho)}{\theta} t}$.
- We restrict the analysis to parameter values such that $\rho - n > \frac{1-\theta}{\theta}[f(1) - (\delta+\rho)]$, for intertemporal utility to take a finite value.

(In red on this page: changes from pages 26-27.)

Social sub-optimality of the competitive equilibrium IV

- We can then rewrite the differential equation in $\dot{k}_t$ as

$$\dot{k}_t = [f(1) - (n+\delta)]k_t - c_0 e^{\frac{f(1)-(\delta+\rho)}{\theta}t}.$$

- Then, rearranging the terms and multiplying by $e^{-[f(1)-(n+\delta)]t}$,

$$\left\{\dot{k}_t - [f(1) - (n+\delta)]k_t\right\} e^{-[f(1)-(n+\delta)]t} = -c_0 e^{-\varphi^p t},$$

where $\varphi^p \equiv \frac{\theta-1}{\theta} f(1) - (n+\delta) + \frac{\delta+\rho}{\theta}$.

- From the condition $\rho - n > \frac{1-\theta}{\theta}[f(1) - (\delta+\rho)]$, we deduce that $\varphi^p > 0$.

Social sub-optimality of the competitive equilibrium V

- We can therefore integrate the previous equation to get

$$k_t e^{-[f(1)-(n+\delta)]t} - k_0 = \frac{c_0}{\varphi^p} e^{-\varphi^p t} - \frac{c_0}{\varphi^p}$$

$$\text{and then } k_t = \left(k_0 - \frac{c_0}{\varphi^p}\right) e^{[f(1)-(n+\delta)]t} + \frac{c_0}{\varphi^p} e^{\frac{f(1)-(\delta+\rho)}{\theta} t}.$$

- We then rewrite the transversality condition as

$$\lim_{t \to +\infty} \left\{ k_0 - \frac{c_0}{\varphi^p} + \frac{c_0}{\varphi^p} e^{-\varphi^p t} \right\} = 0,$$

which implies that $c_0 = \varphi^p k_0 > 0$ since $\varphi^p > 0$ (as in Chapter 2, c_0 is chosen so as to satisfy the transversality condition).

Social sub-optimality of the competitive equilibrium VI

- We therefore finally obtain

$$k_t = k_0 e^{\frac{f(1)-(\delta+\rho)}{\theta}t},\ c_t = \varphi^p k_0 e^{\frac{f(1)-(\delta+\rho)}{\theta}t} \text{ and } y_t = f(1) k_0 e^{\frac{f(1)-(\delta+\rho)}{\theta}t}.$$

- These results differ from the previous ones, so **the competitive equilibrium is not socially optimal**.
- More precisely, **the competitive equilibrium is socially sub-optimal**: U_0 takes a value strictly lower in the competitive equilibrium than with the $\mathcal{BOOP}$.
- This last result, which can be easily checked with computations, comes from the fact that the $\mathcal{BOOP}$ does not choose the competitive-equilibrium allocation even though this allocation satisfies the three constraints of their optimization problem.

Social sub-optimality of the competitive equilibrium VII

- The growth rate of k_t, c_t and y_t is equal to
  - $\frac{f(1)-(\delta+\rho)}{\theta}$ with the $\mathcal{BOOP}$,
  - $\frac{f'(1)-(\delta+\rho)}{\theta}$ in the competitive equilibrium.
- Now, because of the externality, the marginal social product of capital, $f(1)$, is strictly higher than the marginal private product of capital, $f'(1)$.
- So, **growth is higher with the $\mathcal{BOOP}$**: the latter, who internalizes the externality, orders more investment.
- And, as a consequence, c_0 is lower with the $\mathcal{BOOP}$:

$$\varphi^p k_0 = \left[\varphi - \frac{f(1) - f'(1)}{\theta}\right] k_0 < \varphi k_0.$$

Role of economic policy I

- The social sub-optimality of the competitive equilibrium gives a role to economic policy.

- Part 4 of the tutorials shows that a fiscal authority can implement the $\mathcal{BOOP}$'s allocation in a decentralized way by
  - **subsidizing investment** at a rate such that the private return of capital is equal to its social return,
  - **financing this subsidy with a lump-sum tax** on households, which does not "distort" their choices (lump-sum tax $\equiv$ tax such that the amount that an individual has to pay does not depend on their actions),

  or else, alternatively, by
  - **subsidizing financial incomes** at a rate such that the private return of capital is equal to its social return,
  - **financing this subsidy with labor-income tax**, which does not "distort" households' choices because of the exogenous nature of labor supply.

Role of economic policy II

- In the case of a **positive externality** (like knowledge diffusion), such a **subsidy** system, financed in a lump-sum way, makes private agents **internalize** the **social benefit** of their actions.

- In the case of a **negative externality** (like pollution), a similar system of **taxes**, redistributed in a lump-sum way, makes private agents **internalize** the **social cost** of their actions.

- These taxes/subsidies are called **Pigouvian** taxes/subsidies.

- **Arthur C. Pigou**: English economist, born in 1877 in Ryde, deceased in 1959 in Cambridge, professor at the University of Cambridge from 1896.

Conclusion

Main predictions of the model

- In the short and long terms,
  - growth depends on parameters governing technology, preferences, the dynamics of capital, and only on these parameters,
  - the six stylised facts of Kaldor (1961) are obtained.
- The effect of capital accumulation on growth does not vanish in the long term, thanks to the constant social returns of capital.
- There is neither absolute convergence, nor conditional convergence, of the per-capita-output levels (in logarithm) across countries.
- The competitive equilibrium is socially sub-optimal because of the presence of an externality.
- Economic policies, in the form of Pigouvian subsidies, can implement the socially optimal equilibrium.

Two limitations of the model

- The model corresponds to the special case in which the social returns of capital are constant because the learning-by-doing and knowledge-diffusion effects *exactly* offset the decreasing nature of the private returns of capital (if the social returns of capital were not constant, then the positive implications of the model would be very different).

  $\hookrightarrow$ Chapter 5 does not make any “knife-edge” assumption about the value of a parameter.

- The model explains long-term growth by the **involuntary and non-remunerated accumulation of knowledge**.

  $\hookrightarrow$ Chapter 5 explains it by the voluntary and remunerated accumulation of knowledge, based on the notion of patents.

Appendix

Proof that $\varphi > f(1) - f'(1) > 0$

- We show that $\varphi > f(1) - f'(1) > 0$ in four steps:

1. Differentiating $F(1,x) = xf(\frac{1}{x})$ with respect to $x \in \mathbb{R}^+$, we get $F_2(1,x) = f(\frac{1}{x}) - \frac{1}{x}f'(\frac{1}{x})$. Now $F_2(1,1) > 0$, so $f(1) - f'(1) > 0$.

2. Using $\varphi \equiv f(1) - (n+\delta) - \frac{f'(1)-(\delta+\rho)}{\theta}$, we get $\varphi - [f(1) - f'(1)] = \frac{\theta-1}{\theta}f'(1) - (n+\delta) + \frac{\delta+\rho}{\theta}$.

3. We rewrite the condition $\rho - n > \frac{1-\theta}{\theta}[f'(1) - (\delta+\rho)]$ as $\frac{\theta-1}{\theta}f'(1) > n - \rho + \frac{\theta-1}{\theta}(\delta+\rho)$.

4. We deduce from the previous two steps that $\varphi - [f(1) - f'(1)] > n - \rho + \frac{\theta-1}{\theta}(\delta+\rho) - n - \delta + \frac{\delta+\rho}{\theta} = 0$.